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LETTER TO THE EDITOR

Three-loop calculation of the anomalous dimension of diffusivity for a model of a random walk in a potential random field with long-range correlations

S É Derkachov†, J Honkonen‡ and Yu M Pis'mak†

† Department of Theoretical Physics, State University of Leningrad, Ul'yanovskaya 1, Staryi Petergof, 198904 Leningrad, USSR

‡ Research Institute for Theoretical Physics, University of Helinski, Siltavuorenpenger 20 C, SF-00170 Helsinki, Finland

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Abstract. The three-loop contribution to the anomalous dimension of the diffusion coefficient of the model of a random walk in a potential random field with long-range correlations is calculated. Contrary to earlier conjectures, the result is not zero for logarithmic growth of the correlations, but vanishes only in one and two dimensions, in which the one-loop contribution yields the exact value of the anomalous dimension.

The problem of diffusion in a random (velocity) field has attracted considerable attention [1-3], and a great deal of its properties have been thoroughly studied (see the recent review [4] for references). The case of the potential random velocity field is of particular interest: the renormalisation-group beta function, which governs the long-distance asymptotic behaviour of the model, turns out to be trivial [5] (i.e. all the loop contributions to it vanish) leading to disorder-dependent asymptotic behaviour at the upper critical dimension $d = d_c$, and strong-disorder regime below it $d < d_c$. Moreover, in the two-dimensional case with logarithmically growing correlations of the random potential, two-loop and higher-order contributions to the anomalous dimension vanish [6], i.e. it can be calculated perturbatively exactly. It has been suggested [6, 7] that this is true for arbitrary dimension of space. This conjecture is supported by two-loop calculations [8]. In this letter we show, however, by explicit calculation that the three-loop contribution to the anomalous dimension of the diffusivity does not vanish identically. We show that, apart from the two-dimensional case, the anomalous dimension is determined by the one-loop contribution also in the exactly solvable one-dimensional case with logarithmic growth of the correlations of the random potential.

The problem of diffusion in a potential random velocity field is described by the equation

$$[\partial_t - D_0\nabla(\nabla\psi(x)+\nabla)]c(x, t) = 0 \tag{1}$$

where c is the density of diffusing particles, D_0 is the bare (not renormalised) diffusion coefficient, and ψ is the random potential with zero mean and the correlation function defined by the Fourier transformation F of the (generalised, when necessary) function $1/|\boldsymbol{p}|^{2+2\alpha}$: $C_0(\boldsymbol{x}-\boldsymbol{x}) \equiv \langle\psi(\boldsymbol{x})\psi(\boldsymbol{x}')\rangle_0 = g_0(2\pi)^{-d}F[1/|\boldsymbol{p}|^{2+2\alpha}](\boldsymbol{x}-\boldsymbol{x}')$, i.e.

$$C_0(\boldsymbol{x}-\boldsymbol{x}') = \begin{cases} \dfrac{2g_0 \ln|\boldsymbol{x}-\boldsymbol{x}')}{(4\pi)^{1+\alpha}\Gamma(q+\alpha)} & \text{if } d=2+2\alpha \\ \dfrac{g_0\Gamma[(d/2)-1-\alpha]}{(4\pi)^{d/2}\Gamma(1+\alpha)}\left(\dfrac{2}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)^{d-2-2\alpha} & \text{otherwise} \end{cases} \tag{2}$$

Here, Γ is the gamma function and the (non-negative) bare coupling constant g_0 describes the strength of the disorder. We have omitted the finite additive constant, which may be present in the relations (2) for $d \leqslant 2+2\alpha$. Since the potential ψ enters in the equation (1) with a derivative, this constant is irrelevant.

We shall calculate the retarded Green function of the equation (1) averaged over the random potential ψ. Excluding the variable t by Fourier transformation and using functional representation for the ψ average and the Green function G_ω in the frequency space, we obtain the relation $\langle G_\omega(\boldsymbol{x},\boldsymbol{x}')\rangle_0 = D_0^{-1}G_{\varphi\tilde{\varphi}}(\boldsymbol{x},\boldsymbol{x}')$ where $G_{\varphi\tilde{\varphi}}$ is the full propagator of the field theory defined by the action

$$S = -\tfrac{1}{2}\psi C_0^{-1}\psi + \tilde{\psi}[m_0 + \nabla(\nabla + \nabla\psi)]\varphi. \tag{3}$$

Here, $m_0 = \mathrm{i}\omega/D_0$, and the convention that all closed loops of bare $\varphi\tilde{\varphi}$ propagators are zero is used [6]. All the necessary sums and integrals are implied in the expression (3), and subsequent similar formulae. The field theory (3) is multiplicatively renormalisable and, moreover, it can be renormalised by a single renormalisation constant Z [5]. Introducing the scaling parameter μ we therefore write the renormalised action in the form

$$S_{\mathrm{R}} = -\frac{1}{2\mu^{\varepsilon}}\psi C^{-1}\psi + \tilde{\psi}[m + Z\nabla(\nabla + \nabla\psi)]\varphi \tag{4}$$

where the 'renormalised' correlation function C is obtained from the bare one C_0 by the substitution $g_0 \to g$. The renormalisation constant Z determines the anomalous dimension γ_{D} of the diffusion coefficient $\gamma_{\mathrm{D}} = -\mu\partial \ln Z/\partial\mu|_0$ where the subscript indicates that the partial derivative is taken with fixed values of the bare parameters. As a consequence of the triviality of the beta function of the model (3), its asymptotic behaviour is not universal at the upper critical dimension: the renormalised coupling constant g remains a free parameter, on which the anomalous dimension γ_{D} depends, and we choose $g = g_0$.

Using the general fact that the anomalous dimension and the beta function do not depend on the renormalised mass m of the model [9], we calculate them in the massless theory, and henceforth set $m = 0$. In this case, the normalisation conditions of Green functions are usually defined at some finite values of external momenta, which then determine the momentum scale μ of the renormalised theory. However, we shall be constructing the perturbation expansion for a slightly modified model, for which this procedure is not sufficient, and therefore introduce the scaling parameter μ as the infrared cut-off in the regularised $\psi\psi$ correlation function:

$$C_{\mathrm{reg}}(\boldsymbol{x}-\boldsymbol{x}') = g\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d}\frac{\exp[\mathrm{i}\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')]}{(\boldsymbol{p}^2+\mu^2)^{1+\alpha}}. \tag{5}$$

To remove large-momentum divergences, we use dimensional regularisation of the field theory (3) with the parameter $\varepsilon = 2+2\alpha-d$, where d is the space dimension of the model (the upper critical dimension is $d_c = 2+2\alpha$). The full propagator G of the renormalised massless field theory (4) may be found by averaging the solution $G_\psi(x, y)$ of the equation

$$Z\nabla[\nabla + \nabla\psi(x)]G_\psi(x, y) = -\delta(x-y) \tag{6}$$

over the renormalised distribution of the random potential ψ:

$$G(x-y) = \langle G_\psi(x, y)\rangle \equiv \int \mathrm{D}\psi \exp\left(\frac{1}{2\mu^\varepsilon}\psi C^{-1}\psi\right)G_\psi(x, y).$$

Introducing the function $R(x, y; \psi) = Z_R^{-1}Z\exp[\psi(x)]G_\psi(x, y)$, where Z_R is a new renormalisation constant, and the function $V(x; \psi) = \exp[-\psi(x)]-1$ we obtain from (6) the equation

$$[\nabla^2 + \nabla V(x; \psi)\nabla]R(x, y; \psi) = -Z_R^{-1}\delta(x-y). \tag{7}$$

Following the argument of [6] it can be shown that the averaged solution $\langle R\rangle$ of this equation can be made finite by a suitable choice of the renormalisation constant Z_R, and that the relation (we use the same notation for functions and their Fourier transforms)

$$p^2G(p) = Z^{-1}Z_R\exp(\tfrac{1}{2}C_{\mathrm{reg}}(0))H(p/\mu; g) \tag{8}$$

holds. Here, H is a finite function, and since G is the renormalised full propagator of the field theory (4), we obtain from (8)

$$Z = Z_R\exp(\tfrac{1}{2}C_{\mathrm{reg}}(0)).$$

up to finite renormalisation.

It is convenient to introduce the matrix $\mathbf{T}$ (note that this definition of $\mathbf{T}$ is slightly different from that in [6]) defined by

$$R(p, q; \psi) \equiv \frac{p_m}{p^2}T_{mn}(p, q)\frac{q_n}{q^2} \tag{9}$$

for which we obtain from (7) the equation

$$T_{mn}(p, q) + \int\frac{\mathrm{d}k}{(2\pi)^d}V(p-k)\frac{k_mk_l}{k^2}T_{ln}(k, q) = -(2\pi)^d\delta(p+q)\delta_{mn}Z_R^{-1}.$$

Substituting $P^{\|} = (1-S)/2$, where $S_{mn}(p) \equiv \delta_{mn} - 2p_mp_n/p^2$, for the longitudinal projection operator $P^{\|}_{mn}(p) \equiv p_mp_n/p^2$, and applying the convolution theorem we obtain the equation

$$\begin{aligned}T_{mn}(p, q) &+ \int\frac{\mathrm{d}k}{(2\pi)^d}U(p-k)S_{ml}(k)T_{ln}(k, q)\\ &= -Z_R^{-1}\bigg(2(2\pi)^d\delta(p+q)\frac{q_mq_n}{q^2}\\ &\quad + [(2\pi)^d\delta(p+q)\delta_{ml} + U(p+q)S_{ml}(q)]S_{ln}(q)\bigg)\end{aligned}$$

where the Fourier transform of the function $U(x; \psi) \equiv \tanh[\psi(x)/2]$ has been introduced, and the relation $S^2 = 1$ used. Averaging the formal solution of this equation

$$\mathbf{T} = -Z_R^{-1}[S + 2(1+US)^{-1}P^{\|}] \tag{10}$$

over the renormalised distribution of the random potential ψ we obtain a graphical representation of the function $\langle \mathbf{T} \rangle$, constructed from vector lines corresponding to $\boldsymbol{S}$, scalar lines corresponding to the $\psi\psi$ correlator (5) and vertices with two S-legs and any odd number of ψ-legs generated by the function $U = \tanh(\psi/2)$. The advantage of this representation compared with the original field theory (4) is that the number of graphs to be calculated is significantly reduced. The price is that the graphical expression for $\langle \mathbf{T} \rangle$ does not correspond to any multiplicatively renormalisable auxiliary field theory. Therefore, to carry out the renormalisation, we use the fact that the function $\langle R \rangle$ can be made finite by a suitable choice of the renormalisation constant Z_{R}.

To this end, we define the self-energy matrix $\hat{\Sigma}$ of the expression for $\langle \mathbf{T} \rangle$

$$\langle (1 + U\boldsymbol{S})^{-1} \rangle \equiv (1 - \hat{\Sigma}\boldsymbol{S})^{-1} \tag{11}$$

and the function $Q : \langle R(\boldsymbol{p}, \boldsymbol{q}; \psi) \rangle \equiv (2\pi)^d \delta(\boldsymbol{p} + \boldsymbol{q}) Q(\boldsymbol{p})/\boldsymbol{p}^2$ for which we obtain from (9), (10) and (11) the equation

$$Q(\boldsymbol{p}) = Z_{\mathrm{R}}^{-1}[1 - 2\,\mathrm{Tr}((1 - \hat{\Sigma}(\boldsymbol{p})\boldsymbol{S}(\boldsymbol{p}))^{-1} P^{\parallel}(\boldsymbol{p}))].$$

Expanding Z_{R} and $\hat{\Sigma}$ in g

$$Z_{\mathrm{R}} = 1 + Z_1 + Z_2 + Z_3 + \ldots \qquad \hat{\Sigma} = \hat{\Sigma}_1 + \hat{\Sigma}_2 + \hat{\Sigma}_3 + \ldots$$

and choosing the Z_is to ensure the absence of divergences in ε in the function Q to third order in g, we obtain

$$Z_1 = -2\bar{\Sigma}_1$$

$$Z_2 = 2\bar{\Sigma}_1^2 - 2\bar{\Sigma}_2$$

$$Z_3 = -\tfrac{4}{3}\bar{\Sigma}_1^3 + 4\bar{\Sigma}_1\bar{\Sigma}_2 - 2\overline{(\Sigma_3 + \tfrac{1}{3}\Sigma_1^3)}.$$

Here, we have used a kind of minimal subtraction scheme in the sense that the Z_is have been determined by the singular in ε parts of the contributions of the graphs. The bar above a quantity denotes the extraction of the singular part of it. The constants Σ_i are related to the $\hat{\Sigma}$-matrices as follows:

$$\hat{\Sigma}_{i,mn}(\boldsymbol{p}) = \delta_{mn}[\Sigma_i + F_i(\boldsymbol{p}^2)] + p_m p_n J_i(\boldsymbol{p}^2)$$

where $f_i \to 0$ and $J_i < \infty$ in the limit $\boldsymbol{p} \to 0$.

From the earlier two-loop results [8] it follows that $\bar{\Sigma}_2 = 0$, and the Z_is are determined by the one-loop contribution Σ_1 and the three-loop contribution Σ_3. Z_1 is given by a single one-loop graph and is equal to $Z_1 = -\alpha g/(4\pi)^{1+\alpha}(2+\alpha)\varepsilon$ whereas the expression for Σ_3 includes 17 topologically different graphs. In the complicated calculation of these graphs we have used the uniqueness method [10], and also developed new tools, which we feel deserve a separate discussion. Therefore, we do not dwell on the details of the calculation here, and quote only the final result for Z_3:

$$Z_3 = \frac{Z_1^3}{6} + \frac{\alpha(1+2\alpha)g^3}{48(4\pi)^{3+3\alpha}\Gamma(2+\alpha)^3\varepsilon}$$

$$\times\{\Gamma(2+\alpha)^2(\alpha^2 - 3\alpha + 1)h(2+\alpha) + (\alpha^2 - \alpha - 3)[\psi'(1+\alpha) - \psi'(1)] + 2\}$$

where ψ' is the trigamma function, and $h(t)$ is a function expressed through a two-loop massless graph, which we, unfortunately, were not able to calculate in a closed form. The definition of h is

$$h(t) \equiv \frac{\boldsymbol{x}^2}{\pi^{2t}} \int \mathrm{d}\boldsymbol{x}_1\, \mathrm{d}\boldsymbol{x}_2 \frac{1}{\boldsymbol{x}_1^2 \boldsymbol{x}_2^2 (\boldsymbol{x}_1 - \boldsymbol{x}_2)^{2(t-1)} (\boldsymbol{x} - \boldsymbol{x}_2)^2 (\boldsymbol{x} - \boldsymbol{x}_1)^{2(t-1)}}$$

where the dimension of space is $d=2t$. However, the value of this function can be computed for $t=2,3,\ldots$, for example $h(t)=6\zeta(3)\sim 7.212$, and $h(3)=\zeta(3)-\frac{1}{3}\sim 0.8687$, where ζ is the Riemann zeta function. From this it follows, in particular, that the difference $Z_3-Z_1^3/6$ cannot be identically zero. This is sufficient to prove that the higher than one-loop order contributions to the anomalous dimension of the diffusivity γ_D are not identically zero.

For γ_D we obtain

$$\gamma_D=\frac{g}{(4\pi)^{1+\alpha}(2+\alpha)}+\frac{\alpha(1+2\alpha)g^3}{48(4\pi)^{3+3\alpha}\Gamma(2+\alpha)^3}\times\{\Gamma(2+\alpha)^2(\alpha^2-3\alpha+1)h(2+\alpha)+(\alpha^2-\alpha-3)[\psi'(1+\alpha)-\psi'(1)]+2\}. \tag{12}$$

The three-loop contribution vanishes with α as it should and, interestingly enough, it also vanishes at $\alpha=-\frac{1}{2}$, which corresponds to $d_c=1$. Due to infrared divergences, it is not obvious that the results of the field theory (4) hold for $d=1$, but the validity of the formula (12) in one dimension may be justified using the exact solution of the one-dimensional equation

$$Z\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right)G_\psi(x,y)=-\delta(x-y).$$

The fundamental solution of this equation

$$G_\psi(x,y)=-Z^{-1}\int_y^\infty \mathrm{d}\xi\,\exp[\psi(\xi)-\psi(x)].$$

averaged over the random potential distribution yields the averaged Green function $G(x-y)=\langle G_\psi(x,y)\rangle$. Due to fixed dimensionality, we cannot use dimensional regularisation, and therefore take the regularised $\psi\psi$ correlation function in the form

$$C_{\mathrm{reg}}(x-y)\equiv g\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\frac{[1-\theta(|p|-\Lambda)]\exp[p(x-y)]}{(p^2+\mu^2)^{1/2}}.$$

We obtain

$$G(x-y)=-Z^{-1}\int_y^x \mathrm{d}\xi\,\exp[C_{\mathrm{reg}}(0)-C_{\mathrm{reg}}(\xi+x)]$$

which, after taking the limit $\mu\to 0$ and leaving only the divergent in Λ contribution, yields $G(x-y)=-Z^{-1}\Lambda^{g/\pi}A|x-y|^{1+g/\pi}$ where A is a constant independent of Λ. By a suitable choice of the renormalisation constant Z we obtain

$$G(x-y)=-|x-y|^{1+g/\pi}.$$

On the other hand, renormalisation group argument yields $G(\boldsymbol{x}-\boldsymbol{y})\sim|\boldsymbol{x}-\boldsymbol{y}|^{2-d+\gamma_D}$, therefore in one dimension the anomalous dimension $\gamma_D=g/\pi$ exactly, which is in accord with the relation (12).

In conclusion, we have calculated to three-loop order the anomalous dimension of the diffusion coefficient for the model of diffusion in a potential random field and found that, contrary to earlier conjectures, the three-loop contribution does not vanish identically. In addition to the two-dimensional problem with logarithmically growing correlations, we have shown that the anomalous dimension is given exactly by the one-loop expression also in one dimension.

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